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EFFECTIVE RIGIDITIES OF COMPOSITE PLATES*

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A problem of constructing the effective rigidities of composite plates, i.e. of the rigidities ensuring that the solutions of the initial problem of the three-dimensional theory of elasticity for an inhomogeneous body are close to a solution of some problem in the theory of quasihomogeneous plates, is investigated. The homogenization method /1–5/ is used to obtain the solution. The theory of plates employed below has the following significant characteristic feature, namely the fact that mechanical hypotheses are used to execute the passage from the three-dimensional problem of the theory of elasticity to the two-dimensional problem of the theory of plates. The methods used in the passage indicated explain the discrepancies occurring in various methods of constructing the effective characteristics.

1. First we consider the problem of stability of plates satisfying a priori the Kirchhoff–Love hypotheses. The flexure of such plates was studied in /5/. Let us assume for definiteness that an inhomogeneous plate is clamped rigidly and the stresses $\lambda\sigma_{11}$, $\lambda\tau$, $\lambda\sigma_{22}$ act in its plane. The stresses are such, that increase in the value of the parameter λ leads to loss of stability. The equation describing the instant when the plate loses its stability, has the form /6/

$$\begin{gathered}L_\varepsilon w \equiv [D(w_{,11}+\nu w_{,22})]_{,11}+2[D(1-\nu)w_{,12}]_{,12}+\\ [D(w_{,22}+\nu w_{,11})]_{,22}=\lambda[(\sigma_{11}w_{,1})_{,1}+\\ (\tau w_{,2})_{,1}+(\tau w_{,1})_{,2}+(\sigma_{22}w_{,2})_{,2}]\equiv\lambda M_\varepsilon w\end{gathered} \tag{1.1}$$

The elastic constants D and ν depend on the parameter $\varepsilon \ll 1$ (absolute or relative size of the inhomogeneities). In many cases this relationship has the form $D, \nu = D, \nu\,(\bar{x}/\varepsilon)$ /1–5/. The stresses σ_{11}, τ, σ_{22} in the plane of the plate are obtained from the solution of the plane problem of the theory of elasticity for an inhomogeneous body, and also represent rapidly oscillating functions which depend on the parameter ε. We shall study the asymptotic behavior of the smallest in modulo eigenvalue of the problem (1.1), (i.e. the smallest critical load) as $\varepsilon \to 0$.

Note 1. Problems of averaging the differential operators were considered by a number of authors, e.g. in /1–5/. In particular, /7,8/ dealt with the certain problems of averaging of spectra. The results of the above papers imply that the averaging, which is regarded in mechanical sense as a comparison of the composite material with another homogeneous material with nearly identical mechanical behavior, has a corresponding mathematical concept of G-limit of the operators /9,4/, or of the Γ-, G-limit of the functionals /7,9/. In particular, when the mass forces are arbitrary, the displacements of the initial and averaged bodies are nearly equal if and only if the averaged body uses the coefficients of the G-limit operator as the elastic characteristics (this follows, after slight modifications, from /9/). The author of /10/ notes that under the above conditions the elastic deformation energies of the initial and averaged body are nearly equal, and this justifies the restrictions imposed below on the operators of the problem (1.1).

The problem (1.1) is conveniently studied using an abstract formulation. Let the following linear, self-conjugate and bounded uniformly in ε operators be given:

$$\begin{gathered}L_\varepsilon, L: H_2^{0m}(Q)\to H_2^{-m}(Q)\quad m,k\in R\\ M_\varepsilon, M: H_2^{0k}(Q)\to H_2^{-k}(Q)\quad k<m\end{gathered} \tag{1.2}$$

Let also a number $c>0$ exist such that for every φ belonging to $H_2^{0m}(Q)$ we have $\langle L_\varepsilon\varphi,\varphi\rangle_m \geqslant c\,\|\varphi\|_m^2$ ($\langle,\rangle_n$, $\|\ \|_n$ denotes the pairing operation and norm in $H_2^{0n}(Q)$) and $M_\varepsilon \not\equiv 0$.

Proposition 1. If the sequence of operators L_ε G-converges to operator L (its definition is given in /4/) and for every $\varphi\in H_2^{0k}(Q)$ we have $M_\varepsilon\varphi\to M\varphi$ in $H_2^{-m}(Q)$ with $M\not\equiv 0$, then $|\lambda_\varepsilon|\to|\lambda_0|$. Here λ_ε is the smallest in modulo eigenvalue of the problem

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$$L_\varepsilon f = \lambda M_\varepsilon f \tag{1.3}$$

and λ_0 is the smallest in modulo eigenvalue of the problem

$$Lf = \lambda M f \tag{1.4}$$

Proof. Consider the following problem on the space $H_2^{0k}(Q)$:

$$f = \lambda L_\varepsilon^{-1} M_\varepsilon f \tag{1.5}$$

We shall show that (1.5) and (1.3) are equivalent. According to /11/, when conditions (1.2) hold, then the operator $L_\varepsilon^{-1} : H_2^{-m}(Q) \to H_2^{0m}(Q)$ exists and is bounded uniformly in ε. Therefore, from the inclusion $H_2^{0m}(Q) \subset H_2^{0k}(Q)$, $H_2^{-k}(Q) \subset H_2^{-m}(Q)$ and the standard inequalities connecting the norms of these spaces it follows that the operator L_ε^{-1} is defined as an operator acting from $H_2^{-k}(Q)$ to $H_2^{0k}(Q)$ and is also uniformly bounded in ε /11/. Then the problem (1.5) is formulated correctly and therefore equivalent to (1.3).

Let us denote by $\{f_\varepsilon\}$ the eigenfunctions corresponding to the smallest in modulo eigenvalues (clearly, they can differ from each other only in sign). By virtue of the uniform boundedness in ε of the operators M_ε (condition (1.2)) and L_ε (see above), we have the following estimate for $|\lambda_\varepsilon|$

$$|\lambda_\varepsilon^{-1}| = \frac{\| L_\varepsilon^{-1} M_\varepsilon f_\varepsilon \|_k}{\| f_\varepsilon \|_k} \leqslant \frac{\| L_\varepsilon^{-1} \| \| M_\varepsilon \| \| f_\varepsilon \|_k}{\| f_\varepsilon \|_k} \leqslant K < \infty \tag{1.6}$$

with the constant K independent of ε. The first equality in (1.6) defines the eigenvalue of the problem (1.3). We further introduce a family of functions $\{g_\varepsilon\} = \{\lambda_\varepsilon^{-1} f_\varepsilon : \| f_\varepsilon \|_k = 1\}$. The last equality in the formula for $\{g_\varepsilon\}$ can always be obtained by normalizing the eigenfunctions. By virtue of the definition of $\{g_\varepsilon\}$ we have $g_\varepsilon = L_\varepsilon^{-1} M_\varepsilon f_\varepsilon$ and from this we find, using the fact that the operator L_ε^{-1}, as the operator acting from $H_2^{-k}(Q)$ into $H_2^{0m}(Q)$, is bounded uniformly in ε /11/, that $\| g_\varepsilon \|_m \leqslant K_1 < \infty$ with constant K_1 independent of ε. Then the sequence $\{g_\varepsilon\}$ will be weakly compact in $H_2^{0m}(Q)$ /11/ and by virtue of the fact that $k < m$, compact in $H_2^{0k}(Q)$ /12/. The sequence $\{\lambda_\varepsilon^{-1}\}$ will be compact in R by virtue of (1.6). Then we can select from the sequence $\{\lambda_\varepsilon^{-1}, f_\varepsilon\}$ a converging sequence $\{\lambda_{\varepsilon'}^{-1}, f_{\varepsilon'}\}$. Let $\lambda_{\varepsilon'}^{-1} \to \mu_0$, $g_{\varepsilon'} \to g_0$ in $H_2^{0k}(Q)$. By definition of $\lambda_\varepsilon, g_\varepsilon$ we have

$$\lambda_{\varepsilon'}^{-1} g_{\varepsilon'} = L_{\varepsilon'}^{-1} M_{\varepsilon'} g_{\varepsilon'} \tag{1.7}$$

From the uniform in ε continuity of the operators M_ε we find that $M_{\varepsilon'}(g_{\varepsilon'} - g_0) \to 0$ in $H_2^{-k}(Q) \subset H_2^{-m}(Q)$ and $M_{\varepsilon'} g_0 \to M g_0$ in $H_2^{-m}(Q)$ by virtue of the proposition, therefore $M_{\varepsilon'} g_{\varepsilon'} \to M g_0$ in $H_2^{-m}(Q)$. Then by virtue of the uniform in ε, continuity of L_ε^{-1} as the operator acting from $H_2^{-m}(Q)$ to $H_2^{0k}(Q)$ (see above) and the G-convergence of the sequence of operators L_ε and L, we obtain the equation for

$$\mu_0 g_0 = L^{-1} M g_0 \tag{1.8}$$

Next we shall show that $g_0 \neq 0$. Since $\| g_0 \|_k = \lim \| g_{\varepsilon'} \|_k = \lim \| f_{\varepsilon'}/\lambda_{\varepsilon'} \|_k = \lim \lambda_{\varepsilon'}^{-1}$, it is sufficient to show that $\mu_0 \neq 0$. The operators L_ε and M_ε are selfconjugate, therefore $L_\varepsilon^{-1} M_\varepsilon$ is also selfconjugate and we have for it a variational principle, /11,12/ according to which

$$|\lambda_{\varepsilon'}^{-1}| \geqslant \frac{|(L_{\varepsilon'}^{-1} M_{\varepsilon'} \varphi, \varphi)_k|}{\| \varphi \|_k^2}, \quad \forall \varphi \in H_2^{0k}(Q)$$

Now, passing to the limit at fixed φ we obtain

$$|\mu_0| \geqslant \frac{|(L^{-1} M \varphi, \varphi)_k|}{\| \varphi \|_k^2}, \quad \forall \varphi \in H_2^{0k}(Q) \tag{1.9}$$

Since L^{-1}, $M \not\equiv 0$, it follows from (1.9) that $\mu_0 \neq 0$. Let us now put $\lambda_0 = \mu_0^{-1}$ and $f_0 = \lambda_0 g_0$. Clearly, $f_{\varepsilon'} \to f_0$ in $H_2^{0k}(Q)$. From (1.8) and $g_0 \neq 0$, $\mu_0 \neq 0$ it follows that λ_0 and f_0 are, respectively, the eigenvalue and the eigenfunction of the problem (1.4). Moreover,

$$\lambda_{\varepsilon'}^{-1} \to \mu_0 = \lambda_0^{-1} \tag{1.10}$$

By virtue of (1.9) λ_0 is the eigenvalue smallest in modulo. There can be not more than two such numbers (disregarding the multiplicities) and they can differ from each other only in sign. It follows therefore that the sequence $\{\lambda_\varepsilon\}$ can have not more than two limit points equal in modulo, and this implies, with the compactness of $\{\lambda_\varepsilon\}$ taken into account, that $|\lambda_\varepsilon| \to |\lambda_0|$.

Note 2. The result stating that only the moduli of the least critical loads converge cannot, in general, be improved. We can illustrate this using the problem of stability of a plate under shear, with a spectrum symmetrical about the zero. In the case of homogeneous plates the result can be sharpened just by considering the class of subcritical states with positivie definite subcritical stress tensors $\sigma_{11}, \tau, \sigma_{22}$. In the composite plates however, a subcritical state of general form occurs, and the above method becomes inapplicable. Below we shall show that the sufficient condition for convergence of the smallest critical loads is, that the limiting subcritical state described by the operator M by sign-definite.

Proposition 2. If, under the conditions of proposition 1, the operator M is sign-definite, then $\lambda_\varepsilon \to \lambda_0$.

Proof. According to (1.9), (1.10) any converging subsequence of the sequence $\{\lambda_\varepsilon, f_\varepsilon\}$ can have, as the limit of the first argument, only the eigenvalue smallest in modulo. In the case when M is sign-definite, such a numer is unique (disregarding the multiplicity) and this implies, by virtue of the compactness of $\{\lambda_\varepsilon\}$, as was shown in the proof of proposition 1, that $\lambda_\varepsilon \to \lambda_0$.

Let us now confirm that the operators defined by (1.1) satisfy the conditions (1.2) when we put $m = 2$ and $k > 1.5$. When $\infty > C \geqslant D$, $1 - \nu \geqslant C > 0$, the operator L_ε is positive definite /13/, bounded uniformly in ε, and obviously self-conjugate. Consider the operator M_ε. We have

$$|\langle M_\varepsilon f, \varphi\rangle_1| = \Big|\iint\limits_Q \sigma_{ij} f_{,i} \varphi_{,j}\, d\overline{x}\Big| \leqslant \tag{1.11}$$
$$|\,\|\sigma_{ij}\|_0\,|\,\|\nabla f\|_{L_4}\|\nabla\varphi\|_{L_4} \leqslant |\,\|\sigma_{ij}\|_0\,|\,\|f\|_k\|\varphi\|_k$$

The last two inequalities follow from the continuity of the inclusion of $H_2^{0m}(Q)$ in $H_4^{01}(Q)$ at $n \geqslant 1.5$ /12/. This implies that for $k \geqslant 1.5$ the operator M_ε defined by (1.1) is bounded uniformly in ε, provided that the following condition holds:

$$|\,\|\sigma_{ij}\|_0\,| \leqslant K_2 < \infty \tag{1.12}$$

where the constant K_2 is independent of ε. Moreover, the operator M_ε is clearly self-conjugate.

Note 3. We are dealing with the case when the subcritical stresses $\sigma_{11}, \tau, \sigma_{22}$ in the plane of the plate are found from the solution of the first boundary value problem of the plane theory of elasticity for an inhomogeneous body

$$A_\varepsilon \overline{u}_\varepsilon = 0 \text{ or } \lambda F \quad \sigma_{ij}^{\ \varepsilon} = a_{ijkl}^\varepsilon (\operatorname{def} \bar{u}_\varepsilon)_{kl} \tag{1.13}$$
$$\overline{u}_\varepsilon|_{\partial Q} = \lambda \bar{u}^\circ \text{ or } 0$$

The problem (1.13) was studied in /1–3/, and their results show that, in particular, the stresses which are the a_{ijkl}^ε-gradients of the solution /4/ converge weakly in $L_2(Q)$ to the stresses σ_{ij} determined from the solution of the problem strongly G-limited with respect to (1.13). This enables us to construct the operator M, and we note that this implies that the condition (1.12) is fulfilled.

Proposition 3. If $\sigma_{11} = \sigma_{11}^\varepsilon$, $\tau = \sigma_{12}^\varepsilon = \sigma_{21}^\varepsilon$, $\sigma_{22} = \sigma_{22}^\varepsilon$, then the operator M is

$$M = \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial}{\partial x_j}\right)$$

Proof. We have for $f \in H_2^{0k}(Q) \subset H_4^{01}(Q)$

$$\|M_\varepsilon f - Mf\|_{-2} = \sup_{\|\varphi\|_2 \leqslant 1} |((\sigma_{ij}^{\ \varepsilon} - \sigma_{ij}) f_{,i}, \varphi_{,j})_0| \tag{1.14}$$

Let us introduce the set $F = \{\overline{\psi} : \overline{\psi} = \nabla\varphi, \|\varphi\|_2 \leqslant 1\} \subset L_4(Q)$. By virtue of the inclusion theorem /12/ F is compact in $L_q(Q)$ for any $q < \infty$. The equation (1.14) can be rewritten in the form

$$\|M_\varepsilon f - Mf\|_{-2} = \sup_{\overline{\psi} \in F} |((\sigma_{ij}^{\ \varepsilon} - \sigma_{ij}), f_{,j}\psi_j)_0| \tag{1.15}$$

We introduce into F a finite "δ-mesh" /11/, which is possible since F is compact in $L_4(Q)$. Since $\sigma_{ij}^\varepsilon \to \sigma_{ij}$ weakly in $L_2(Q)$, we find that the right-hand side of (1.15) tends to zero as $\varepsilon \to 0$ on the δ-mesh. The continuity of the operators M_ε uniform in ε now implies the convergence to zero on the whole set F, and this completes the proof.

Following the same procedure we can deal with the case of a hinged and mixed (rigid and hinged) support of a plate. In the first case the solution of (1.1) is sought on the set V representing the closure of the set $\{f \in C^\infty(Q) : f(\overline{x}) = 0, \ \overline{x} \in dQ\}$ on the norm $H_2^2(Q)$ /13/. Here the operator L_ε is positive definite uniformly in ε, provided that $\infty > C \geqslant D$, $1 - \nu \geqslant c > 0$ which can be proved following /13/. From this it follows that all previous results can be transferred to this case without any changes, and the same applies to the case of mixed support. We note that in the case of a hinged support of a plate the rapidly oscillating elastic constants appear in the boundary conditions, namely in the condition for the moments. In this case the boundary condition becomes averaged, i.e. if the limit solution is smooth, then the boundary condition for it is obtained by integration by parts in the equation $\langle Lw, \varphi\rangle_2 = \lambda \langle Mw, \varphi\rangle_1$ obtained from the variational principle, for every $\varphi \in H_2^{02}(Q)$. We see here that the same boundary conditions can be obtained by integrating by parts the equation $\langle Lw, \varphi\rangle_2 = 0$, since the expression $\langle Mw, \varphi\rangle_1$ does not yield on Γ integrals with respect to ∂Q.

The above argumentation was carried out for plates of constant thickness h occupying a constant region Q. The absolute dimensions of the inhomogeneities tend here to zero, and this case is characteristic for the composite materials /5/.

In a number of problems the study of the relative sizes of inhomogeneities is of interest. Let us denote by T and t the characteristic dimensions of the plate and inhomogeneities, respectively. We pass to the dimensionless variables using the following formulas (a prime denotes a dimensionless variable):

$$\bar{x}' = \bar{x}/T, \quad h' = h/T^{1/2}, \quad w' = w, \quad \bar{u}_\varepsilon' = \bar{u}_\varepsilon, \; u^{0\prime} = \bar{u}^0 \tag{1.16}$$

It can be confirmed that equations (1.13) of the plane theory of elasticty without the mass forces, and equation (1.1) with the condition of rigid support, both contain T only in the coefficients after the change of variables (1.16). Moreover, if in the initial variables $E, \nu = E^0, \nu^0\,(\bar{x}/t)$, then in the new variables $E, \nu = E^0, \nu^0(\bar{x}/(t/T))$ where E^0 and ν^0 denote the functions in which the characteristic dimension of the change is equal to unity. In this case the relative size of the inhomogeneities $\varepsilon = t/T$ can be taken as the small parameter.

The above remark makes it possible to apply the propositions 1 and 2 to the ribbed plates for which the condition of smallness of the absolute rib size leads to contradiction with the hypotheses used in the course of deriving (1.1) /6/. The relative smallness of the rib sizes on the other hand, makes it possible to carry out the averaging without contradicting the hypotheses of the plate theory /6/ under the condition that the rib width is of the order of the characteristic thickness of the plate.

Note 4. Although the G-limits exist for a wide class of operators (e.g. for those with rapidly oscillating periodic coefficients) /1 − 5/, the analytic expressions for the G-limits of the operators of the form (1.1), (1.13) have been obtained up to now only for the case of uniform distribution of the inhomogeneities /5,4/.

Example. We consider a rectangular plate reinforced with a system of ribs parallel to the Ox_2 axis, and hinged. In this case we have $D, \nu = D, \nu(x_1/\varepsilon)$; $Q = [0, M) \times [0, L]$. Let the subcritical state of stress in the plane of the plate be determined from the solution of (1.13) with the boundary condition $\sigma_{ij}^\varepsilon n_j = \sigma_i^\circ$ on ∂Q. We shall show below that the proposition 3 will remain, in this case, in force (see also the analysis of the problem of the plate theory in /5/). The limit state of stress is $\sigma_{ij} = \sigma_i^\circ \delta_{ij}$ and we can establish that

$$|\lambda_0| = \pi^2 \left\{ \frac{\alpha}{L^4} + 2\,\frac{\alpha\langle\nu\rangle + \langle D(1-\nu)\rangle}{M^2L^2} + \frac{\langle D\rangle - \langle D\nu^2\rangle + \langle D\nu\rangle\langle\nu\rangle}{M^4} \right\} \left\{ \frac{\sigma_1^\circ}{L^2} + \frac{\sigma_2^\circ}{M^2} \right\}^{-1}$$

Here $\alpha = \langle D^{-1}\rangle^{-1}$ and $\langle\ \rangle$ denotes the averaging over a period of the functions D and ν are periodic, otherwise $\langle\ \rangle$ denotes the operation of taking a weak limit in $L_2(Q)$. When $\nu = \mathrm{const}$, the formula simplifies and assumes the form

$$|\lambda_0| = \pi^2 \langle D\rangle \left\{ \left(\frac{1}{L^2} + \frac{1}{M^2}\right)^2 - \left(1 - \frac{\alpha}{\langle D\rangle}\right)\left(\frac{1}{L^2} + \frac{2\nu}{M^2}\right)\frac{1}{L^2} \right\} \times \left\{ \frac{\sigma_1^\circ}{L^2} + \frac{\sigma_2^\circ}{M^2} \right\}^{-1}$$

2. Next we consider the problem of determining effective characteristics of the plates essentially inhomogeneous across their thickness, to which the Kirchhoff—Love hypotheses cannot a priori be applied. In this case we formulate the problem as follows: to construct, from the A_ε-operator of the theory of elasticity of an inhomogeneous body (with mixed boundary conditions), an operator L of the theory of quasihomogeneous plates such that the solution of the first problem (theory of elasticity for an inhomogeneous body) coincides, as $\varepsilon \to 0$, with the solution of the second problem (theory of plates) with some prescribed accuracy. We shall show that the above problem can be solved by introducing an intermediate averaging step according to the diagram

$$\begin{array}{ccccc} A_\varepsilon & \to & A & \to & L \\ \downarrow & & \downarrow & & \downarrow \\ \bar{u}_\varepsilon & \to & \bar{u} & \to & w \end{array}$$

The passage $A_\varepsilon \to A$ consists of calculating the G-limit of the sequence of functionals corresponding to the sequence of operators A_ε /9/, and A is an operator corresponding to the G-limiting functional. We denote by $\bar{\mathbf{u}}$ the solution of the G-limit problem. The passage $A \to L$ is executed as follows. As we shall show below, the operator A exists for a wide class of inhomogeneities and has the form (2.1) in which the coefficients a^ε_{ijkl} are replaced by certain homogenized coefficients a_{ijkl} possessing the properties of elastic constants (symmetry and positive definiteness) /14/. The coefficients are independent of ε and can therefore be regarded as elastic constants of some quasihomogeneous elastic material. We proceed as follows: beginning with the mechanical characteristics of the material corresponding to the homogenized coefficients a_{ijkl}, we choose a kinematic model and use the methods of /6,15,14/ to construct a problem of the theory of plates with some operator L. The solution w of this problem approximates $\bar{\mathbf{u}}$ in $L_2(Q)$ with required accuracy $\alpha(h, R)$ (h is the thickness and R is the characteristic dimension of the plates in the plane).

In order to confirm that the operator L is the required one, it is sufficient to show that $\bar{\mathbf{u}}_\varepsilon \to \bar{\mathbf{u}}$ in $L_2(Q)$ as $\varepsilon \to 0$, and, that the operator A has indeed the properties assigned to it. Consider the initial problem describing the composite plate as a solid inhomogeneous body

$$[a^\varepsilon_{ijkl}\, u^k_{\varepsilon,l}]_{,j} = f_i(x) \tag{2.1}$$

$$\bar{\mathbf{u}}_\varepsilon = 0 \ \text{ on } \Gamma_u = \partial Q_1 \times [-h, h]/2 \tag{2.2}$$

$$\bar{a}^\varepsilon_{ijkl}\,(\operatorname{def} \bar{\mathbf{u}}_\varepsilon)_{kl} n_j = 0 \ \text{ on } \Gamma_6 = \{x \in Q : x_3 = \pm h/2\} \tag{2.3}$$

We assume that the elastic constants $a^\varepsilon_{ijkl} \in L_\infty(Q)$ and that positive bounded constants C and c exist such that $C\,|\,e_{ij}\,|^2 \geqslant a^\varepsilon_{ijkl} e_{ij} e_{kl} \geqslant c\,|\,e_{ij}\,|^2$ for all ε and $e_{ij} = e_{ji}$. The solution of (2.1) — (2.3) is sought on V-closure of the norm $H_2^1(Q)$ of the set of functions belonging to $C^\infty(Q)$ vanishing near the part Γ_u of the boundary /13/. We shall show that the families of solutions of the problems (2.1)— (2.3) have, generally speaking, limit points.

Lemma. The family $\{\bar{\mathbf{u}}_\varepsilon\}$ is weakly compact in V.

Proof. It is sufficient to show that $\{\bar{\mathbf{u}}_\varepsilon\}$ is bounded in V, and to show this is sufficient, in turn, to confirm (see e.g. /13/) that for the boundary conditions (2.2), (2.3) the Korn inequality holds, the latter implying the required proof. We shall show that any limit point of the sequence $\{\bar{u}_\varepsilon\}$ is a solution of the G-limit problem. With this in mind, we note that the problem (2.1) — (2.3) is equivalent to the problem of minimization ($\langle , \rangle$ denotes the pairing operation in V)

$$\bar{\mathbf{u}}_\varepsilon \in V : J_\varepsilon(\bar{\mathbf{v}}) + \langle \bar{f}, \bar{\mathbf{v}} \rangle \to \min, \quad \bar{\mathbf{v}} \in V \tag{2.4}$$

$$J_\varepsilon(\bar{\mathbf{v}}) = \frac{1}{2} \int\limits_Q a^\varepsilon_{ijkl}\,(\operatorname{def} \bar{\mathbf{v}})_{ij}\,(\operatorname{def} \bar{\mathbf{v}})_{kl}\, d\bar{\mathbf{x}}$$

Let us give one test for G-convergence of the functionals in the Marcellini /9/ sense. A sequence of functionals J_ε G-converges to the functional J if and only if:

1°. For every $\bar{\mathbf{v}} \in V$ there exists a sequence $\{\bar{\mathbf{v}}_\varepsilon\} \subset V$ such that $\bar{\mathbf{v}}_\varepsilon \to \bar{\mathbf{v}}$ weakly in V and $J_\varepsilon(\bar{\mathbf{v}}_\varepsilon) \to J(\bar{\mathbf{v}})$.

2°. For every sequence $\{\bar{\mathbf{v}}_\varepsilon\} \subset V$ such that $\bar{\mathbf{v}}_\varepsilon \to \bar{\mathbf{v}}$ weakly in V, $J(\bar{\mathbf{v}}) \leqslant \underline{\lim}\, J_\varepsilon(\bar{\mathbf{v}}_\varepsilon)$.

Proposition 4. If $\bar{\mathbf{u}}$ is a limit point of the family $\{\bar{\mathbf{u}}_\varepsilon\}$, then $\bar{\mathbf{u}}$ solves the problem of minimization of the functional $J(\bar{\mathbf{v}}) + \langle \bar{f}, \bar{\mathbf{v}} \rangle$ on V

Proof. Let $\bar{\mathbf{u}}_\varepsilon \to \bar{\mathbf{u}}$ weakly in V. The set V is weakly closed (by virtue of the closure and convexity /12/), therefore $\bar{\mathbf{u}}$ belongs to V. Using the feature of G-convergence given above, we can write the following sequence of relationships: for any $\bar{\mathbf{v}} \in V$

$$J(\bar{\mathbf{v}}) + \langle \bar{\mathbf{f}}, \bar{\mathbf{v}} \rangle = \lim\,[J_\varepsilon(\bar{\mathbf{v}}_\varepsilon) + \langle \bar{\mathbf{f}}, \bar{\mathbf{v}}_\varepsilon \rangle] \geqslant \underline{\lim} \min_{\bar{\mathbf{v}} \in V} [J_\varepsilon(\bar{\mathbf{v}}) + \langle \bar{\mathbf{f}}, \bar{\mathbf{v}} \rangle] = \lim\,[J_\varepsilon(\bar{\mathbf{u}}_\varepsilon) + \langle \bar{\mathbf{f}}, \bar{\mathbf{u}}_\varepsilon \rangle] \geqslant J(\bar{\mathbf{u}}) + \langle \bar{\mathbf{f}}, \bar{\mathbf{u}} \rangle$$

where $\{\bar{\mathbf{v}}_\varepsilon\}$ is a sequence determined for the given element $\bar{\mathbf{v}}$ by the condition 1°. The above inequality yields the proof.

Next we shall show that the G-limiting operator A exists for a wide class of distributions of inhomogeneities, and has the properties listed above (i.e. it is an operator of the theory of elasticity of a quasihomogeneous body). We do this by showing that if the sequence of operators A_ε with the domain of definition $\bar{\varphi} + H_2^{01}(Q)$ has a G-limit for any $\bar{\varphi} \in V$, then it has a G-limit when regarded as a sequence of operators defined on V, and these G-limits coincide. This will solve our problem, since the G-limit of the operators on $\bar{\varphi} + H_2^{01}(Q)$ exists and has the required properties for a wide class of distributions of the inhomogeneities (e.g. when conditions of the type N of /3/ hold, see also /1—5/).

Proposition 5. If the sequence of functionals J_ε of the form (2.4) with the domain of definition $\bar{\varphi} + H_2^{01}(Q)$ G-converges, for any $\bar{\varphi}$ belonging to V, to the functional J, then the same sequence J_ε defined on V G-converges to the functional J (also defined on V).

Proof. We shall confirm that the test for G-convergence of the functionals /9/ is fulfilled for the functionals J_ε and J defined on V. Let $\{\bar{u}_\varepsilon\}$ be a family of solutions of (2.4), and $\bar{u}$ a solution of the problem obtained from (2.4) when J_ε is replaced by J. We note that since the functional J is defined on $\bar{\varphi} + H_2^{01}(Q)$ for all $\bar{\varphi}$ of V, it is also defined on V. First we show that $\underline{\lim}\, J_\varepsilon(\bar{u}_\varepsilon) \geqslant J(\bar{u})$. To do this we introduce $\bar{U}_\varepsilon$ as the solution of the problem of minimization

$$\bar{U}_\varepsilon \in \bar{u} + H_2^{01}(Q) : J_\varepsilon(\bar{v}) + \langle \bar{f}, \bar{v}\rangle \to \min, \quad \bar{v} \in \bar{u} + H_2^{01}(Q);\ \bar{f} \in V^*$$

By virtue of the inclusion $\bar{u} + H_2^{01}(Q) \subset V$ the problem is formulated correctly, both on $u + H_2^{01}(Q)$ and on V. We have

$$J_\varepsilon(\bar{u}_\varepsilon) = J_\varepsilon(\bar{U}_\varepsilon) + A_\varepsilon(\bar{U}_\varepsilon, \bar{u}_\varepsilon - \bar{U}_\varepsilon) + J_\varepsilon(\bar{u}_\varepsilon - \bar{U}_\varepsilon) \tag{2.5}$$

$$A_\varepsilon(\bar{\varphi}, \bar{\psi}) = \int_Q a_{ijkl}^\varepsilon (\det \bar{\varphi})_{ij} (\operatorname{def} \bar{\psi})_{kl}\, d\bar{x}$$

We shall show that the second term in the right-hand side of (2.5) tends to zero as $\varepsilon \to 0$. By definition of $\bar{u}_\varepsilon$ we find that for any $\bar{\varphi}$ of V, and in particular for $\bar{U}_\varepsilon \in V$

$$A_\varepsilon(\bar{u}_\varepsilon, \bar{U}_\varepsilon) = -\langle \bar{f}, \bar{U}_\varepsilon\rangle$$

From this we obtain, using the fact that A_ε is symmetric,

$$A_\varepsilon(\bar{U}_\varepsilon, \bar{u}_\varepsilon - \bar{U}_\varepsilon) = A_\varepsilon(\bar{U}_\varepsilon, \bar{u}_\varepsilon) - A_\varepsilon(\bar{U}_\varepsilon, \bar{U}_\varepsilon) = -\langle \bar{f}, \bar{u}_\varepsilon\rangle - A_\varepsilon(\bar{U}_\varepsilon, \bar{U}_\varepsilon) \tag{2.6}$$

Further, by virtue of the G-convergence of J_ε and J on $\bar{u} + H_2^{01}(Q)$ /7/, we have

$$\bar{U}_\varepsilon \to \bar{u} \text{ weakly in } V,\ J_\varepsilon(\bar{U}_\varepsilon) \to J(\bar{u}) \tag{2.7}$$

The last limit of (2.7) implies $A_\varepsilon(\bar{U}_\varepsilon, \bar{U}_\varepsilon) \to A(\bar{u}, \bar{u})$ (here we have used the fact that the operator A has the form (2.4) with coefficients a_{ijkl}; the form of A is determined in the same manner as that of A_ε). Applying (2.7) to (2.6) we obtain the required relationship, since, what was said before implies that the limit of the right-hand side of (2.6) is equal to $\langle \bar{f}, \bar{u}\rangle - A(\bar{u}, \bar{u}) = 0$. The last equation follows from the fact that $\bar{u}$ is a solution of the G-limiting problem.

Passing now in (2.5) to the lower limit and taking into account (2.7), we obtain

$$\underline{\lim}\, J_\varepsilon(\bar{u}_\varepsilon) \geqslant \underline{\lim}\, J_\varepsilon(\bar{U}_\varepsilon) = J(\bar{u})$$

which yields, with the help of the Euler equation,

$$\underline{\lim}\, [J_\varepsilon(\bar{u}_\varepsilon) + \langle \bar{f}, \bar{u}_\varepsilon\rangle] \geqslant J(\bar{u}) + \langle \bar{f}, \bar{u}\rangle \tag{2.8}$$

On the other hand, since $\bar{U}_\varepsilon$ belongs to V, we have

$$J(\bar{u}_\varepsilon) + \langle \bar{f}, \bar{u}_\varepsilon\rangle \leqslant J_\varepsilon(\bar{U}_\varepsilon) + \langle \bar{f}, \bar{U}_\varepsilon\rangle \tag{2.9}$$

Passing in (2.9) with help of (2.7) to the upper limit and combining the resulting inequality with (2.8), we arrive at the relation

$$\lim\, [J_\varepsilon(\bar{u}_\varepsilon) + \langle \bar{f}, \bar{u}_\varepsilon\rangle] = J(\bar{u}) + \langle \bar{f}, \bar{u}\rangle$$

for any $\bar{f}$ belonging to V^*. The above relation represents the above-mentioned test for the G-convergence of functionals /9/.

Note 5. Proposition 5 substantiates the example discussed above (with help of the results of /5/).

Note 6. The strongest demand in proposition 5 is, that the G-limits of the sequence of the functionals J_ε defined on $\bar{\varphi} + H_2^{01}(Q)$ be independent of the function $\bar{\varphi}$. The condition is fulfilled if the coefficients a_{ijkl}^ε satisfy the conditions of /3/ embracing a wide class of distribution of the inhomogeneities (in particular the cases of periodic and quasiperiodic reinforcement/1,3/).

Note 7. The case when a part of the plate Γ_σ is acted upon by surface forces $\bar{g}$, is dealt with completely analogously. If the function $\bar{g}$ is smooth, then the G-limited problem is obtained by replacing the coefficients a_{ijkl}^ε in (2.1) — (2.3) by a_{ijkl}.

Corollary. When applying NOte 1 to plates, we can interpret it as follows: let a passage from a composite plate to a homogeneous plate described by the operator L' be executed in some unspecified manner. The necessary and sufficient condition for the flexures or elastic deformation energies of the initial composite and averaged plate to coincide with the accuracy of the order $\alpha(h, R)$ as $\varepsilon \to 0$, under the arbitrary mass forces is, that the operator L' coincides with the operator L constructed above, with the accuracy of up to the terms correcting the solution, of the order $\alpha(h, R)$.

Note 8. In comparing $\bar{u}_\varepsilon$ and $\bar{u}$ with w, we regard w as the displacements reestablished according to the solution of the problem of plate theory $w(\bar{x}), \bar{x} \in Q_1$ over the whole region Q, and based on the kinematic hypotheses used.

Example. As we have already shown, analytic expressions for the coefficients of the G-limited operators /5/ are known for materials with one-dimensional distribution of the inhomogeneities. This enables us to carry out, using the results obtained, a full investigation of the multilayer composite plates. We shall quote the results, obtained by direct application of the methods given in the present paper.

For the effective flexural rigidity D of a multilayer plate made of isotropic components, we have the formula

$$D = h^3E/[12(1-\nu^2)] \equiv (h^3/12)\langle E_\varepsilon/(1-\nu_\varepsilon^2)\rangle$$

where $E_\varepsilon, \nu_\varepsilon$ are the Young's modulus and Poisson's ratio of the components, and E, ν are the corresponding homogenized characteristics. In the case of plates composed of materials with large differences in the values of their moduli, the ratio of the homogenized shear modulus G and E is small, and the passage $A \to L$ should employ the models which take into account the transverse shear of the plate. The coefficients accompanying the functions describing the transverse shear are proportional to h^5 in the framework of the models of /15/, and can have a quantity of the order of D (proportional to h^3) even in the case of thin plates ($h \ll 1$).

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